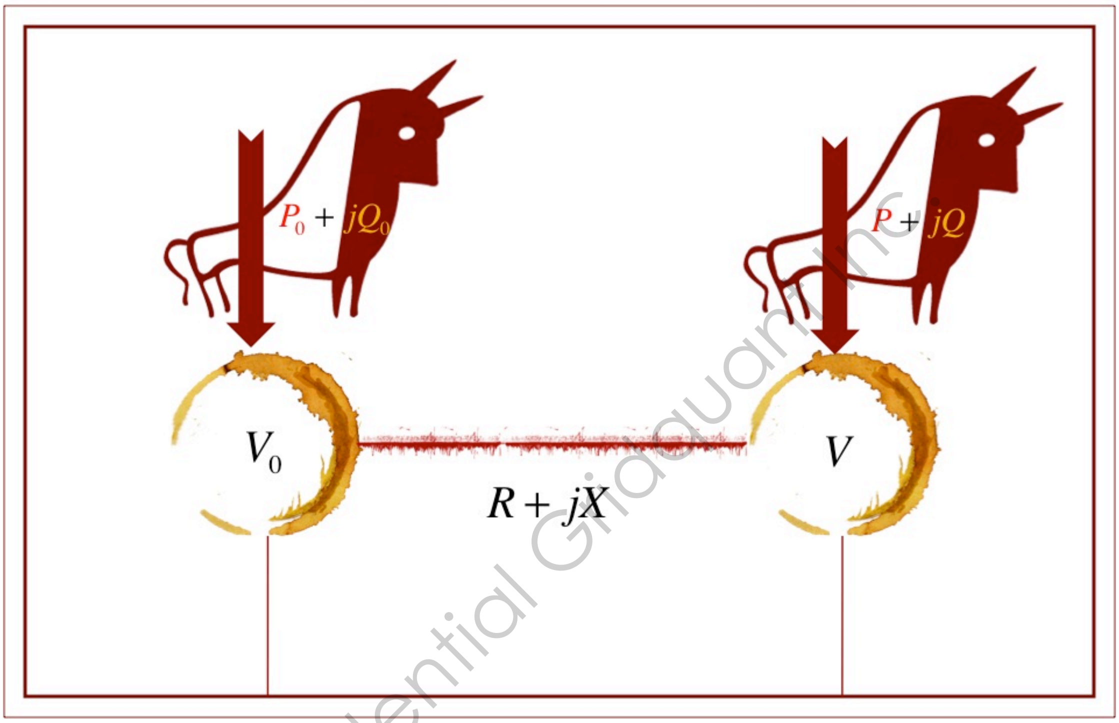
Exact Developments

CONFIDENTIAL

Two Bus Loadflow



NOTES

Exact Two Bus Network

Introduction

Power Flow

One of the buses is the swing. Its voltage is fixed to a complex value: V_0. The unknown is the second bus voltage V. This may be obtained using Ohm's law:

$$\frac{V - V_0}{Z} = I$$

Unfortunately this simple equation gets some complication from the fact that some non linear being is taking (or giving) a constant complex power S from the second bus , so one has to use instead:

$$S = VI^* \quad ; \quad I = \frac{S^*}{V^*}$$

The loadflow equation is then:

$$V = V_0 + Z\frac{S^*}{V^*} \quad ; \quad Z = R + jX \quad ; \quad S = P + jQ$$

Adimensional Variables

It is convenient to introduce:

$$U \equiv \frac{V}{V_0}$$

which verifies

$$U = 1 + Z\frac{S^*}{V_0 V_0^* U^*}$$

The complex magnitude that determines the outcome is the combination of line, swing and complex power parameters given by:

$$\sigma \equiv \frac{ZS^*}{|V_0|^2} \quad ; \quad \sigma_R = \frac{XQ + RP}{|V_0|^2} \quad ; \quad \sigma_I = \frac{XP - RQ}{|V_0|^2}$$

In most practical cases one has:

$$X \gg R$$

and the real part of σ is basically proportional to reactive power while the imaginary part is proportional to active power. This is a simplifying physical image that can be helpful to keep in mind although, as R growths, the active and reactive power roles in σ get progressively mixed up.

For the adimensional voltage the loadflow equation is:

$$U = 1 + \frac{\sigma}{U^*}$$

Exact Solution

Let us start by finding the exact solution obtained from the conventional algebraic setting:

$$|U|^2 = U^* + \sigma$$

obtained by multiplying both sides by U^* which leads to an equivalent equation if $U \neq 0$. Separating real and imaginary parts:

$$|U|^2 - U_R = \sigma_R$$
$$U_I = \sigma_I$$

The second equation is explicit and the first becomes a second degree algebraic one:

$$U_R^2 + \sigma_I^2 = U_R + \sigma_R \quad ; \quad U_R^2 - U_R - \sigma_R + \sigma_I^2 = 0 \quad ; \quad (U_R - \frac{1}{2})^2 = \frac{1}{4} + \sigma_R - \sigma_I^2$$

One has to take a square root. Therefore we have "two" solutions:

$$U_R = \frac{1}{2} \pm \sqrt{\frac{1}{4} + \sigma_R - \sigma_I^2} \quad ; \quad U_I = \sigma_I$$

subject to the condition:

$$D \equiv \frac{1}{4} + \sigma_R - \sigma_I^2 \geq 0$$

otherwise there is no solution whatsoever.

Voltage Colapse

In this last case, the combination of line constants, power settings and fixed swing voltage, as combined in σ, exceed the maximum power deliverable by the circuit and the problem is impossible. The limiting state is given by:

$$\frac{1}{4} + \sigma_R - \sigma_I^2 = 0$$

In this state the adimensional voltage U has the single solution 1/2. The two "branches" meet at this value. This is the voltage collapse limit.

Branch Selection

If D > 0 one has two values for U. However we are describing a physical system, so one of them has to be discarded. There is only a way to do this: Study the behaviour of U as σ changes. This seems to be only an insubstantial remark. Quite the contrary is true. We are changing a parameter equation into a functional one with the voltage as a function of the parameters. In other words,it is necessary to consider U(σ) as the relevant object to have a satisfactory model for the physics behind.

Functional dependence

The behaviour of U(σ) can be summarized as follows. If one takes the plus sign in the case of zero charge one obtains U=1 i.e.: voltage equals to swing and there is no intensity flowing. When the state approaches the collapse, voltage gets lower and lower until it reaches the limit point U=1/2 with the maximum possible flow coming from the swing bus. In any state, if one tries to improve voltage by lowering the reactive load or rising the reactive generation (positive rise of reactive power), one obtains the desired output since $|U|$ growths with σ_R .

If one takes the minus sign, everything gets reversed. In the no load state one has U=0 and the flow from the swing bus is maximum. All of it goes to losses since no power is demanded. As the state approaches the collapse, things "improve". Voltage rises and flow diminishes until we reach the limit point U=1/2 where it matches the other solution. In this curious branch, if one tries to low charge or rise generation to improve voltage, one gets the opposite result since voltage drops when positive reactive injection rises. If there are demons trying to keep voltage and power into limits, this state will quickly and unavoidably tend to U=0. It is an unstable solution from the dynamic point of view.

Then Behaviour selects the acceptable solution. In other words, the loadflow problem is about functions U(σ) not about numbers U. Sometimes we will call this the extended loadflow problem to emphasize the functional character of the variables.

Holomorphic Embedding

Within this view, it is clearer if we introduce a complex charge parameter s, and perform everywhere the formal replacement:

$$\sigma \rightarrow s\sigma$$

Then σ will retain its role as a fixed parameter and the functional dependence will be through s. Of course, the original problem is recovered when s = 1. The equation becomes

$$U(s) = 1 + \frac{s\sigma}{\bar{U}(s)}$$

with

$$\bar{U}(s) \equiv U(s^*)$$

which is the correct holomorphic extension of the original parameter U^* representing the complex conjugate of the voltage. The loadflow equation is then changed into a pair of functional equations

$$U(s) = 1 + \frac{s\sigma}{\bar{U}(s)}$$
$$\bar{U}(s) = 1 + \frac{s\sigma^*}{U(s)}$$

involving two holomorphic functions $U, \bar{U}$ of the complex variable s. This will be referred as the holomorphically embedded loadflow equation system. The system can easily be solved to obtain

$$U(s) = \frac{1}{2} \pm \sqrt{\frac{1}{4} + s\sigma_R - s^2\sigma_I^2 + js\sigma_I}$$

which obviouly reproduces our former solution at $s = 1$. Indeed it povides useful additional information reflected in its analytic structure.

Continued Fractions

By far, the best way of solving the loadflow equation:

$$U(s) = 1 + \frac{s\sigma}{\bar{U}(s)}$$

is the following one: Don't solve it !!. The equation itself is telling explicitely what the solution is in "continued fraction terms". Just iterate the above expression to get

$$U(s) = 1 + \cfrac{s\sigma}{1 + \cfrac{s\sigma^*}{1 + \cfrac{s\sigma}{1 + \cfrac{s\sigma^*}{1 + \cfrac{s\sigma}{1 + \ldots}}}}}$$

The denominator goes under the rug at infinity. As we shall see the unstable branch goes to the same place.

Partial quotients

If one stops the process after N iterations one obtains the N-th partial quotient: $U_N(s)$. It is well known that its explicit form is

$$U_N(s) = \frac{A_N(s)}{B_N(s)}$$

$$A_N(s) = A_{N-1}(s) + a_N A_{N-1}(s) \quad ; \quad A_{-1}(s) = 1 \quad ; \quad A_0(s) = 1$$
$$B_N(s) = B_{N-1}(s) + a_N B_{N-1}(s) \quad ; \quad B_{-1}(s) = 0 \quad ; \quad B_0(s) = 1$$

$$a_{2n+1} = s\sigma \quad ; \quad n = 0, 1, \ldots$$
$$a_{2n} = s\sigma^* \quad ; \quad n = 1, 2, \ldots$$

Three term recursion relation

It is natural here to separate even and odd convergents:

$$A_{2n+1}(s) = A_n^{(-)}(s) \quad ; \quad B_{2n+1}(s) = B_n^{(-)}(s)$$
$$A_{2n}(s) = A_n^{(+)}(s) \quad ; \quad B_{2n}(s) = B_n^{(+)}(s)$$

Then one may substitute the odd part in terms of the even one:

$$A_n^{(-)}(s) = A_n^{(+)}(s) + s\sigma A_{n-1}^{(-)} \quad ; \quad B_n^{(-)}(s) = B_n^{(+)}(s) + s\sigma B_{n-1}^{(-)}$$
$$A_{n+1}^{(+)}(s) = A_n^{(-)}(s) + s\sigma^* A_n^{(+)} \quad ; \quad B_{n+1}^{(+)}(s) = B_n^{(-)}(s) + s\sigma^* B_n^{(+)}$$

and obtain the closed three term recursion:

$$A_{n+1}^{(+)}(s) - (1 + 2s\sigma_R)A_n^{(+)} + s^2 |\sigma|^2 \; A_{n-1}^{(+)} = 0$$
$$B_{n+1}^{(+)}(s) - (1 + 2s\sigma_R)B_n^{(+)} + s^2 |\sigma|^2 \; B_{n-1}^{(+)} = 0$$

$$A_n^{(-)}(s) = A_{n+1}^{(+)}(s) - s\sigma^* A_n^{(+)}$$
$$B_n^{(-)}(s) = B_{n+1}^{(+)}(s) - s\sigma^* B_n^{(+)}$$

which is solved as usual by finding the rootsof the characteristic polynomial

$$\lambda^2 - (1 + 2s\sigma_R)\lambda + s^2(\sigma_R^2 + \sigma_I^2) = 0$$
$$\lambda_\pm = \frac{1}{2} + s\sigma_R \pm \sqrt{\frac{1}{4} + s\sigma_R - s^2\sigma_I^2}$$

and adjusting the constants in the general solution by means of the "initial conditions" of the general recursion relation among numerators and denominators of the continued fraction convergents:

$$A_n^{(+)} = C_+\lambda_+^n + C_-\lambda_-^n$$
$$A_0 = A_0^{(+)} = C_+ + C_- = 1$$
$$A_{-1} = A_{-1}^{(-)} = A_0^{(+)} - s\sigma * A_{-1}^{(+)} = 1 - s\sigma * \left(C_+\lambda_+^{-1} + C_-\lambda_-^{-1}\right) = 1$$
$$C_+\lambda_+^{-1} + (1 - C_+)\lambda_-^{-1} = 0$$
$$C_+ = \frac{-\lambda_-^{-1}}{\lambda_+^{-1} - \lambda_-^{-1}} = \frac{\lambda_+}{\lambda_+ - \lambda_-} \quad ; \quad C_- = -\frac{\lambda_-}{\lambda_+ - \lambda_-}$$

so one obtains for the numerators the explicit form

$$A_n^{(+)} = \frac{\lambda_+^{n+1} - \lambda_-^{n+1}}{\lambda_+ - \lambda_-}$$

Let us do the same for the denominators:

$$
\begin{aligned}
&B_n^{(+)} = D_+\lambda_+^n + D_-\lambda_-^n \\
&B_0 = B_0^{(+)} = D_+ + D_- = 1 \\
&B_{-1} = B_{-1}^{(-)} = B_0^{(+)} - s\sigma * B_{-1}^{(+)} = 1 - s\sigma * \left(D_+\lambda_+^{-1} + D_-\lambda_-^{-1}\right) = 0 \\
&D_+\lambda_+^{-1} + (1-D_+)\lambda_-^{-1} = \frac{1}{s\sigma^*} \\
&D_+ = \frac{-\lambda_-^{-1} + \dfrac{1}{s\sigma^*}}{\lambda_+^{-1} - \lambda_-^{-1}} = \frac{\lambda_+ - \dfrac{\lambda_+\lambda_-}{s\sigma^*}}{\lambda_+ - \lambda_-} = \frac{\lambda_+ - \dfrac{s^2\sigma\sigma^*}{s\sigma^*}}{\lambda_+ - \lambda_-} = \frac{\lambda_+ - s\sigma}{\lambda_+ - \lambda_-} \quad D_- = -\frac{\lambda_- - s\sigma}{\lambda_+ - \lambda_-}
\end{aligned}
$$

which leads us to

$$
B_+^{(n)} = \frac{\lambda_+^{n+1} - \lambda_-^{n+1}}{\lambda_+ - \lambda_-} - s\sigma\frac{\lambda_+^n - \lambda_-^n}{\lambda_+ - \lambda_-}
$$

Limit of the continued fraction

We have thus obtained then the exact expression for the convergent:

$$
U_n^{(+)}(s) \equiv \frac{A_n^{(+)}(s)}{B_n^{(+)}(s)} = \frac{A_{2n}(s)}{B_{2n}(s)} = \frac{\dfrac{\lambda_+^{n+1} - \lambda_-^{n+1}}{\lambda_+ - \lambda_-}}{\dfrac{\lambda_+^{n+1} - \lambda_-^{n+1}}{\lambda_+ - \lambda_-} - s\sigma\dfrac{\lambda_+^n - \lambda_-^n}{\lambda_+ - \lambda_-}} = \frac{\lambda_+ - \lambda_-(\dfrac{\lambda_-}{\lambda_+})^n}{\lambda_+ - \lambda_-(\dfrac{\lambda_-}{\lambda_+})^n - s\sigma(1-(\dfrac{\lambda_-}{\lambda_+})^n)}
$$

which can be put in the form

$$
U_n^{(+)}(s) = U_{2n}(s) = \frac{\lambda_+ - \lambda_-(\dfrac{\lambda_-}{\lambda_+})^n}{\lambda_+ - s\sigma - (\lambda_- - s\sigma)(\dfrac{\lambda_-}{\lambda_+})^n}
$$

By using the expression for the roots one may write

$$
\lambda_\pm = \frac{1}{2} + s\sigma_R \pm \sqrt{\frac{1}{4} + s\sigma_R - s^2\sigma_I^2} \equiv \frac{1}{2} + s\sigma_R \pm \Delta \quad ; \quad \frac{\lambda_-}{\lambda_+} = \frac{\dfrac{1}{2} + s\sigma_R - \Delta}{\dfrac{1}{2} + s\sigma_R + \Delta}
$$

so one has

$$
0 < \frac{\lambda_-}{\lambda_+} < 1 \quad ; \quad if \quad \frac{1}{4} + s\sigma_R - s^2\sigma_I^2 > 0 \Rightarrow \frac{1}{2} + s\sigma_R > \frac{1}{4} + s^2\sigma_I^2
$$

Therefore one can compute the limit as $n \to \infty$ of the continued fraction partial quotients as

$$\lim_{n\to\infty} U_n^{(+)}(s) = \lim_{n\to\infty} U_{2n}(s) = \frac{\lambda_+}{\lambda_+ - s\sigma} = \frac{\frac{1}{2} + s\sigma_R + \Delta}{\frac{1}{2} + s\sigma_R + \Delta - s\sigma_R - js\sigma_I} =$$

$$\frac{\left(\frac{1}{2} + s\sigma_R + \Delta\right)\left(\frac{1}{2} + \Delta + js\sigma_I\right)}{\left(\frac{1}{2} + \Delta - js\sigma_I\right)\left(\frac{1}{2} + \Delta + js\sigma_I\right)} = \frac{\left(\frac{1}{2} + s\sigma_R + \Delta\right)\left(\frac{1}{2} + \Delta + js\sigma_I\right)}{(\frac{1}{2} + \Delta)^2 + s^2\sigma_I^2} =$$

$$= \frac{\left(\frac{1}{2} + s\sigma_R + \Delta\right)\left(\frac{1}{2} + \Delta + js\sigma_I\right)}{\frac{1}{4} + \Delta + \frac{1}{4} + s\sigma_R - s^2\sigma_I^2 + s^2\sigma_I^2} = \frac{1}{2} + \Delta + js\sigma_I$$

This is the final result

$$\lim_{n\to\infty} U_n^{(+)}(s) = \lim_{n\to\infty} U_{2n}(s) = \frac{1}{2} + \sqrt{\frac{1}{4} + s\sigma_R - s^2\sigma_I^2} + js\sigma_I \quad !$$

Therefore one has recovered the exact voltage value in the limit. Notice that it is necessary for the roots to be real. If the radicand is negative, the roots are a pair of complex conjugate numbers. Therefore the ratio:

$$\frac{\lambda_-}{\lambda_+}$$

is a unit modulus complex number. Then there is no limit as n goes to infinity since one has oscillatory behaviour. It is possible to check that the limiting case $\Delta = 0$, also converges to the right solution although the convergence rate is only linear in this case.

Theorem

In summary one has obtained the best possible result:

If the exact solution exists, the method will find it with the correct plus sign. It is impossible for the method to obtain the unstable solution associated with the other branch.
If the exact solution does not exist, the method does not converge to anything. So there is no possibility of obtaining a spurious solution.

Odd partial quotients

It is straightforward to repeat this calculation for the odd part. The algebraic highlights are:

$$A_n^{(+)} = \frac{\lambda_+^{n+1} - \lambda_-^{n+1}}{\lambda_+ - \lambda_-} \quad ; \quad B_+^{(n)} = \frac{\mu_+ \lambda_+^n - \mu_- \lambda_-^n}{\lambda_+ - \lambda_-}$$

$$\mu_\pm \equiv \lambda_\pm - s\sigma = \frac{1}{2} \pm \Delta - js\sigma_I$$

$$A_n^{(-)} = A_{n+1}^{(+)} - s\sigma^* A_n^{(+)} = \frac{\lambda_+^{n+2} - \lambda_-^{n+2} - s\sigma^*(\lambda_+^{n+1} - \lambda_-^{n+1})}{\lambda_+ - \lambda_-} = \frac{\mu_+^* \lambda_+^{n+1} - \mu_-^* \lambda_-^{n+1}}{\lambda_+ - \lambda_-}$$

$$B_n^{(-)} = B_{n+1}^{(+)} - s\sigma^* B_n^{(+)} = \frac{\mu_+ \lambda_+^{n+1} - \mu_- \lambda_-^{n+1} - s\sigma^*(\mu_+ \lambda_+^n - \mu_- \lambda_-^n)}{\lambda_+ - \lambda_-} =$$

$$\frac{\mu_+(\lambda_+ - s\sigma^*)\lambda_+^n - \mu_-(\lambda_- - s\sigma^*)\lambda_-^n}{\lambda_+ - \lambda_-} = \frac{\mu_+ \mu_+^* \lambda_+^n - \mu_- \mu_-^* \lambda_-^n}{\lambda_+ - \lambda_-}$$

$$\mu_\pm \mu_\pm^* = (\frac{1}{2} \pm \Delta)^2 + s^2\sigma_I^2 = \frac{1}{4} \pm \Delta + \frac{1}{4} + s\sigma_R - s^2\sigma_I^2 + s^2\sigma_I^2 = \lambda_\pm$$

$$B_n^{(-)} = \frac{\lambda_+^{n+1} - \lambda_-^{n+1}}{\lambda_+ - \lambda_-}$$

and then

$$A_n^{(-)}(s) = A_{2n+1}(s) = \frac{\mu_+^* \lambda_+^{n+1} - \mu_-^* \lambda_-^{n+1}}{\lambda_+ - \lambda_-}$$

$$B_n^{(-)}(s) = B_{2n+1}(s) = \frac{\lambda_+^{n+1} - \lambda_-^{n+1}}{\lambda_+ - \lambda_-}$$

$$U_n^{(-)}(s) = U_{2n+1}(s) = \frac{\mu_+^* \lambda_+^{n+1} - \mu_-^* \lambda_-^{n+1}}{\lambda_+^{n+1} - \lambda_-^{n+1}} = \frac{\mu_+^* - \mu_-^* (\frac{\lambda_-}{\lambda_-})^{n+1}}{1 - (\frac{\lambda_-}{\lambda_-})^{n+1}}$$

$$\lim_{n\to\infty} U_{2n+1}(s) = \mu_+^* = \frac{1}{2} + \sqrt{\frac{1}{4} + s\sigma_R - s^2\sigma_I^2} \; + \; js\sigma_I \quad !$$

so the paradiagonal sequence also converges to the acceptable solution.